

Multi-parameter deformed and nonstandard  $Y(gl_M)$  Yangian symmetry  
in a novel class of spin Calogero-Sutherland modelsB. Basu-Mallick<sup>1\*</sup> and Anjan Kundu<sup>2†</sup>,<sup>1</sup>*Theoretical Physics Group,  
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1/AF Bidhan Nagar, Calcutta 700 064, India***Abstract**

It is well known through a recent work of Bernard, Gaudin, Haldane and Pasquier (BGHP) that the usual spin Calogero-Sutherland (CS) model, containing particles with  $M$  internal degrees of freedom, respects the  $Y(gl_M)$  Yangian symmetry. By following and suitably modifying the approach of BGHP, in this article we construct a novel class of spin CS models which exhibit multi-parameter deformed or ‘nonstandard’ variants of  $Y(gl_M)$  Yangian symmetry. An interesting feature of such CS Hamiltonians is that they contain many-body spin dependent interactions, which can be calculated directly from the associated rational solutions of Yang-Baxter equation. Moreover, these spin dependent interactions often lead to ‘anyon like’ representations of permutation algebra on the combined internal space of all particles. We also find out the general forms of conserved quantities as well as Lax pairs for the above mentioned class of spin CS models, and describe the method of constructing their exact wave functions.

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# 1 Introduction

Algebraic structures of (1+1) dimensional quantum integrable systems with long ranged interactions and their close connection with diverse subjects like conformal field theory, matrix models, fractional statistics, quantum Hall effect etc. have attracted intense attention in recent years [1-17]. In particular it is found that, commutation relations between the conserved quantities of well known spin Calogero-Sutherland (CS) model, given by the Hamiltonian

$$H = -\frac{1}{2} \sum_{i=1}^N \left( \frac{\partial}{\partial x_i} \right)^2 + \frac{\pi^2}{L^2} \sum_{i < j} \frac{\beta(\beta + P_{ij})}{\sin^2 \frac{\pi}{L}(x_i - x_j)}, \quad (1.1)$$

where  $\beta$  is a coupling constant and  $P_{ij}$  is the permutation operator interchanging the ‘spins’ of  $i$ -th and  $j$ -th particles, generate the  $Y(gl_M)$  Yangian algebra [3]. This  $Y(gl_M)$  Yangian algebra [18,19] can be defined through the operator valued elements of a  $M \times M$  dimensional monodromy matrix  $T^0(u)$ , which obeys the quantum Yang-Baxter equation (QYBE)

$$R_{00'}(u-v) \left( T^0(u) \otimes \mathbb{1} \right) \left( \mathbb{1} \otimes T^{0'}(v) \right) = \left( \mathbb{1} \otimes T^{0'}(v) \right) \left( T^0(u) \otimes \mathbb{1} \right) R_{00'}(u-v). \quad (1.2)$$

Here  $u$  and  $v$  are spectral parameters and the  $M^2 \times M^2$  dimensional rational  $R(u-v)$  matrix, having usual  $c$ -number valued elements, is taken as

$$R_{00'}(u-v) = (u-v) \mathbb{1} + \beta P_{00'}. \quad (1.3)$$

So the conserved quantities of spin CS model (1.1) yield a realisation of  $T^0(u)$  matrix satisfying this QYBE (1.2). Moreover, the spin CS Hamiltonian (1.1) can be reproduced in a simple way from the quantum determinant associated with such monodromy matrix. This close connection between  $Y(gl_M)$  Yangian algebra and spin CS model (1.1) helps to find out the related orthogonal basis of eigenvectors and might also play an important role in calculating various dynamical correlation functions [20].

However, it is worth noting that there exist a class of rational  $R$  matrices which satisfy the Yang-Baxter equation (YBE)

$$R_{00'}(u-v) R_{00''}(u-w) R_{0'0''}(v-w) = R_{0'0''}(v-w) R_{00''}(u-w) R_{00'}(u-v), \quad (1.4)$$

(here a matrix like  $R_{ab}(u-v)$  acts nontrivially only on the  $a$ -th and  $b$ -th vector spaces) and reduce to the  $R$  matrix (1.3) at some particular limits of related deformation parameters. These generalisations of rational solution (1.3) are interestingly connected with various multi-parameter dependent deformations of  $Y(gl_M)$  Yangian algebra [21-24] and some integrable lattice models with local interactions [25-27]. The general form of such rational solutions, as well as their ‘nonstandard’ variants (which will be explained shortly), might be written as

$$R_{00'}(u-v) = (u-v) Q_{00'} + \beta P_{00'}, \quad (1.5)$$

where  $P_{00'}$  is the usual permutation matrix which interchanges two vectors associated with 0-th and 0'-th auxiliary spaces, and  $Q_{00'}$  is another  $M^2 \times M^2$  dimensional matrix whose elements may depend on deformation parameters. By substituting (1.5) to (1.4) and using the above mentioned property of  $P_{00'}$ , it is easy to check that the  $R$  matrix (1.5) would be a valid solution of YBE, provided the corresponding  $Q$  matrix satisfies only two conditions:

$$Q_{00'} Q_{00''} Q_{0'0''} = Q_{0'0''} Q_{00''} Q_{00'}, \quad Q_{00'} Q_{0'0} = \mathbb{1}. \quad (1.6)$$

Thus, any solution of eqn.(1.6) will give us a rational  $R$  matrix in the form (1.5), which, in turn, can be inserted to QYBE (1.2) for obtaining a possible extension of  $Y(gl_M)$  Yangian algebra. The simplest solution of eqn.(1.6) is evidently given by  $Q_{00'} = \mathbb{1}$ , which reproduces the original  $R$  matrix (1.3) and the standard  $Y(gl_M)$  Yangian algebra. However, in general, a solution of eqn.(1.6) might also depend on a set of continuous deformation parameters like  $\{h_p\}$ . So these parameters would naturally appear in the defining relations of corresponding extended Yangian algebra. Moreover, the solutions of eqn.(1.6) often admit a Taylor series expansion in the form (up to an over all normalisation factor)

$$Q_{00'} = \mathbb{1} + \sum_p h_p Q_{00'}^p + \sum_{p,q} h_p h_q Q_{00'}^{pq} + \dots, \quad (1.7)$$

where the leading term is an identity operator. Consequently the multi-parameter dependent Yangian algebras, generated through such  $Q$  matrices, would reduce to standard  $Y(gl_M)$  algebra at the limit  $h_i \rightarrow 0$  for all  $i$ . Though many mathematical properties

of these multi-parameter deformed  $Y(gl_M)$  Yangians have been studied earlier, the important problem of constructing quantum integrable models with long range interactions which would respect such Yangian symmetries has received little attention till now. So, it should be quite encouraging to enquire whether there exist some new variant of spin CS Hamiltonian (1.1) which would exhibit a deformed  $Y(gl_M)$  Yangian symmetry associated with rational  $R$  matrix (1.5).

Furthermore, one may like to seek answer of the above mentioned problem in a slightly different context when the  $Q$  matrix, which is obtained as a solution of eqn.(1.6), can not be expanded in the form (1.7). Though in many previous works [21-24] only the type of  $Q$  matrices that can be expanded as (1.7) were discussed, in this article we construct some other forms of  $Q$  matrices which do not yield identity operator as the 0-th order term in their power series expansion. Consequently the Yangian algebras, generated through such ‘nonstandard’  $Q$  matrices and corresponding rational solutions (1.5), will not reduce to  $Y(gl_M)$  algebra at the limit  $h_i \rightarrow 0$ . Due to this reason, those Yangians may be called as ‘nonstandard’ variants of  $Y(gl_M)$  Yangian algebra.

In this article our main aim is to develop a rather general framework for constructing a large class of quantum integrable spin CS Hamiltonians, each of which would exhibit an extended (i.e., multi-parameter deformed or ‘nonstandard’ variants of)  $Y(gl_M)$  Yangian symmetry. So in sec.2 we start with the rational  $R$  matrix (1.5), but do not assume any particular form of corresponding  $Q$  matrix, and attempt to construct a spin CS Hamiltonian from the quantum determinant like object of related Yangian algebra. To this end, we closely follow and suitably modify the pioneering approach of ref.3, where a realisation of  $Y(gl_M)$  algebra is obtained through the conserved quantities of usual spin CS Hamiltonian (1.1). Subsequently, we also describe the method of constructing exact wave functions for spin CS models which exhibit the extended  $Y(gl_M)$  Yangian symmetries. Next, in sec.3, we examine the question of quantum integrability for the above mentioned class of spin CS models and write down the general forms of their Lax pairs as well as conserved quantities. Finally, in sec.4, we consider some specific examples of  $Q$  matrices which satisfy the conditions (1.6), and attempt to find out the concrete forms of related

spin CS Hamiltonians, conserved quantities and Lax pairs. Sec.5 is the concluding section.

## 2 Construction of spin CS Hamiltonian with extended $Y(gl_M)$ Yangian symmetry

Here we like to find out the general form of a spin CS Hamiltonian, whose conserved quantities would produce a realisation of extended  $Y(gl_M)$  Yangian algebra associated with the  $R$  matrix (1.5). So, in our discussion in this section, we shall not assume any specific form of the corresponding  $Q$  matrix and only use the fact that it satisfies the two conditions (1.6).

However, for our purpose of constructing the above mentioned spin CS Hamiltonian, it would be convenient to briefly recall the method of generating the monodromy matrix for quantum integrable spin chains which contain only local interactions [28-30]. To obtain the monodromy matrix for such a spin chain, one considers a Lax operator  $L_i^0(u)$  whose matrix elements (operator valued) depend only on the spin variables of  $i$ -th lattice site and satisfy the QYBE given by

$$R_{00'}(u-v) \left( L_i^0(u) \otimes \mathbb{1} \right) \left( \mathbb{1} \otimes L_i^{0'}(v) \right) = \left( \mathbb{1} \otimes L_i^{0'}(v) \right) \left( L_i^0(u) \otimes \mathbb{1} \right) R_{00'}(u-v), \quad (2.1)$$

where  $R_{00'}(u-v)$  is a solution of YBE (1.4). In a similar way, one can associate a Lax operator on every lattice site of the spin chain. The monodromy matrix for this spin chain, containing  $N$  number of lattice sites, is generated by multiplying all these Lax operators on the auxilliary space as

$$T^0(u) = L_N^0(u) L_{N-1}^0(u) \cdots L_i^0(u) \cdots L_1^0(u). \quad (2.2)$$

By applying (2.1) and also using the fact that the spin variables at different lattice sites are commuting operators, it is easy to prove that the monodromy matrix (2.2) would also satisfy QYBE (1.2) for the same  $R_{00'}(u-v)$  matrix appearing in (2.1). So, by multiplying some ‘local’ solutions of QYBE, one can also generate its ‘global’ solution.

Now, for finding out the Lax operator of a spin chain associated with the rational solution (1.5), we follow the standard procedure of treating the second auxiliary space in  $R_{0i}(u - \eta_i) = L_i^0(u)$  matrix as a ‘quantum’ space and write down the corresponding  $L_i^0(u)$  operator as

$$L_i^0(u) = Q_{0i} + \beta \frac{P_{0i}}{u - \eta_i}, \quad (2.3)$$

where  $\eta_i$  is an arbitrary constant. By using eqn.(1.6), one can also directly check that the Lax operator (2.3) and  $R$  matrix (1.5) satisfy the QYBE (2.1). However the Lax operator (2.3) is not a good choice for our present purpose, since its elements do not contain yet any coordinate or momentum variable which can be related to some new type of spin CS model. So we modify this Lax operator in the following way:

$$\hat{L}_i^0(u) = Q_{0i} + \beta \frac{P_{0i}}{u - \hat{D}_i}, \quad (2.4)$$

where  $\hat{D}_i$ s ( $i \in [1, N]$ ), the so called Dunkl operators, are defined as [3,31]

$$\hat{D}_i = z_i \frac{\partial}{\partial z_i} + \beta \sum_{j>i} \theta_{ij} K_{ij} - \beta \sum_{j<i} \theta_{ji} K_{ij}, \quad (2.5)$$

$z_i = e^{\frac{2\pi i}{L} x_i}$ ,  $\theta_{ij} = \frac{z_i}{z_i - z_j}$ , and  $K_{ij}$ ’s are the coordinate exchange operators which obey the relations

$$K_{ij} z_i = z_j K_{ij}, \quad K_{ij} \frac{\partial}{\partial z_i} = \frac{\partial}{\partial z_j} K_{ij}, \quad K_{ij} z_l = z_l K_{ij}, \quad (2.6a)$$

$$K_{ij}^2 = 1, \quad K_{ij} K_{jl} = K_{il} K_{ij} = K_{jl} K_{il}, \quad [K_{ij}, K_{lm}] = 0, \quad (2.6b)$$

$i, j, l, m$  being all different indices. The Lax operator (2.4) may now be related to the  $i$ -th particle, rather than the  $i$ -th lattice site, which moves continuously on a circle (we have assumed an ordering among the particles). Since the Dunkl operators (2.5) do not act on the spin degrees of freedom, it is evident that this new Lax operator would also satisfy the QYBE (2.1) for our choice of rational  $R$ -matrix (1.5). Moreover, by using (2.5) and (2.6), one can check that these Dunkl operators satisfy the standard commutation relations

$$[\hat{D}_i, \hat{D}_j] = 0, \quad [K_{i,i+1}, \hat{D}_k] = 0, \quad (2.7a, b)$$

$$K_{i,i+1} \hat{D}_i - \hat{D}_{i+1} K_{i,i+1} = \beta, \quad [K_{i,i+1}, \Delta_N(u)] = 0, \quad (2.7c, d)$$

where  $k \neq i, i+1$  and  $\Delta_N(u) = \prod_{i=1}^N (u - \hat{D}_i)$ . Now, by applying the relation (2.7a), it is easy to see that the matrix elements of  $\hat{L}_i^0(u)$  would commute with that of  $\hat{L}_j^0(u)$ , when  $i \neq j$ . Consequently, by using (2.2), we can construct a monodromy matrix like

$$\begin{aligned} \hat{T}^0(u) &= \left( Q_{0N} + \beta \frac{P_{0N}}{u - \hat{D}_N} \right) \left( Q_{0,N-1} + \beta \frac{P_{0,N-1}}{u - \hat{D}_{N-1}} \right) \cdots \left( Q_{01} + \beta \frac{P_{01}}{u - \hat{D}_1} \right) \\ &= \{ \Delta_N(u) \}^{-1} \prod_{i=N}^1 \left[ (u - \hat{D}_i) Q_{0i} + \beta P_{0i} \right], \end{aligned} \quad (2.8)$$

which would satisfy the QYBE (1.2) and, therefore, yield a realisation of extended  $Y(gl_M)$  Yangian algebra associated with the rational  $R$ -matrix (1.5).

However, the above constructed monodromy matrix still contains the coordinate exchange operators  $K_{ij}$  which we want to eliminate from our final expression. So we define a projection operator as

$$\Pi^*(K_{ij}) = \tilde{P}_{ij}, \quad (2.9)$$

where  $\tilde{P}_{ij}$ s are some yet undetermined spin dependent operators which would act on the combined internal space of all particles (i.e., on  $\mathcal{F} \equiv \underbrace{C^M \otimes C^M \otimes \cdots \otimes C^M}_N$ ). The projection operator in eqn.(2.9) is defined in the sense that one should replace  $K_{ij}$  by  $\tilde{P}_{ij}$ , only after moving  $K_{ij}$  in the extreme right side of an expression. However, it is expected that  $\Pi^*$  will produce the same result while acting on the l.h.s. and r.h.s. of each equation appearing in (2.6b). By using such consistency conditions, it is easy to prove that  $\tilde{P}_{ij}$ s must yield a representation of following permutation algebra on the space  $\mathcal{F}$ :

$$\mathcal{P}_{ij}^2 = 1, \quad \mathcal{P}_{ij}\mathcal{P}_{jl} = \mathcal{P}_{il}\mathcal{P}_{ij} = \mathcal{P}_{jl}\mathcal{P}_{il}, \quad [\mathcal{P}_{ij}, \mathcal{P}_{lm}] = 0, \quad (2.10)$$

$i, j, l, m$  being all different indices. As it is well known, the above permutation algebra can be generated by the ‘nearest neighbour’ transposition elements  $\mathcal{P}_{i,i+1}$  ( $i \in [1, N-1]$ ), which satisfy the relations

$$\mathcal{P}_{i,i+1}\mathcal{P}_{i+1,i+2}\mathcal{P}_{i,i+1} = \mathcal{P}_{i+1,i+2}\mathcal{P}_{i,i+1}\mathcal{P}_{i+1,i+2}, \quad [\mathcal{P}_{i,i+1}, \mathcal{P}_{k,k+1}] = 0, \quad \mathcal{P}_{i,i+1}^2 = \mathbb{1}, \quad (2.11a, b, c)$$

where  $|i - k| > 1$ . All other ‘non-nearest neighbour’ transposition elements like  $\mathcal{P}_{ij}$  (with  $j - i > 1$ ) can be expressed through these generators as

$$\mathcal{P}_{ij} = (\mathcal{P}_{i,i+1}\mathcal{P}_{i+1,i+2} \cdots \mathcal{P}_{j-2,j-1}) \mathcal{P}_{j-1,j} (\mathcal{P}_{j-2,j-1} \cdots \mathcal{P}_{i+1,i+2}\mathcal{P}_{i,i+1}). \quad (2.12)$$

So, the projection operator  $\Pi^*$  will be completely defined if we specify its action only on  $N - 1$  number of coordinate exchange operators like  $K_{i,i+1}$ .

In this context one may note that, while constructing a realisation of  $Y(gl_M)$  Yangian algebra through the conserved quantities of usual spin CS model (1.1), a projection operator is defined in ref.3 as:  $\Pi(K_{ij}) = P_{ij}$ . Here  $P_{ij}$  is the standard permutation operator which acts on the space  $\mathcal{F}$  as

$$P_{ij} |\alpha_1 \alpha_2 \cdots \alpha_i \cdots \alpha_j \cdots \alpha_N\rangle = |\alpha_1 \alpha_2 \cdots \alpha_j \cdots \alpha_i \cdots \alpha_N\rangle, \quad (2.13)$$

where  $|\alpha_1 \alpha_2 \cdots \alpha_i \cdots \alpha_N\rangle$  (with  $\alpha_i \in [1, M]$ ) represents a particular spin configuration of  $N$  particles. It is obvious that this  $P_{ij}$  produces a representation of the permutation algebra (2.10). However it is already known that, by taking appropriate limits of some braid group representations which also satisfy the Hecke algebra, one can easily construst many other inequivalent representations of permutation algebra (2.10) on the space  $\mathcal{F}$  [32]. So, while defining a projection operator in eqn.(2.9), we have not chosen  $\tilde{P}_{ij} = P_{ij}$  from the very beginning. In fact, our aim in the following is to find out the precise form of this  $\tilde{P}_{ij}$ , by demanding that the projection of monodromy matrix (2.8), i.e.

$$T^0(u) = \Pi^* [\hat{T}^0(u)], \quad (2.14)$$

would also satisfy QYBE (1.2), when the corresponding  $R$  matrix is taken as (1.5). Evidently, this  $T^0(u)$  would give a solution of QYBE if  $\hat{T}^0(u)$  satisfies the condition:  $\Pi^* [\hat{T}^0(u) \hat{T}^0(v)] = \Pi^* [\hat{T}^0(u)] \Pi^* [\hat{T}^0(v)]$ , or, equivalently

$$\Pi^* [K_{i,i+1} \tilde{P}_{i,i+1} \hat{T}^0(u)] = \Pi^* [\hat{T}^0(u)]. \quad (2.15)$$

By inserting the explicit form of  $\hat{T}^0(u)$  (2.8) to the above equation and assuming that  $\tilde{P}_{i,i+1}$  acts nontrivially only on  $i$ -th and  $(i+1)$ -th spin spaces, (2.15) can be simplified as

$$\begin{aligned} & \Pi^* [K_{i,i+1} ((u - \hat{D}_i)Q_{0i} + \beta P_{0i}) ((u - \hat{D}_{i+1})Q_{0,i+1} + \beta P_{0,i+1})] \\ &= \tilde{P}_{i,i+1} \Pi^* [((u - \hat{D}_i)Q_{0i} + \beta P_{0i}) ((u - \hat{D}_{i+1})Q_{0,i+1} + \beta P_{0,i+1})]. \end{aligned} \quad (2.16)$$

Furthermore, by using eqns.(2.7) and (2.9), the condition (2.16) can be finally expressed as

$$\mathcal{A}_1 \Pi^* [(u - \hat{D}_i)(u - \hat{D}_{i+1})] + \beta \mathcal{A}_2 \Pi^*(u - \hat{D}_i) + \beta \mathcal{A}_3 \Pi^*(u - \hat{D}_{i+1}) + \beta^2 \mathcal{A}_4 = 0, \quad (2.17)$$



where

$$\begin{aligned}\mathcal{A}_1 &= [Q_{0i}Q_{0,i+1}, \tilde{P}_{i,i+1}] , \quad \mathcal{A}_2 = P_{0i}Q_{0,i+1}\tilde{P}_{i,i+1} - \tilde{P}_{i,i+1}Q_{0i}P_{0,i+1} , \\ \mathcal{A}_3 &= Q_{0i}P_{0,i+1}\tilde{P}_{i,i+1} - \tilde{P}_{i,i+1}P_{0i}Q_{0,i+1}, \quad \mathcal{A}_4 = [P_{0i}P_{0,i+1}, \tilde{P}_{i,i+1}] + P_{0i}Q_{0,i+1} - Q_{0i}P_{0,i+1} .\end{aligned}$$

Now, it is immensely interesting to observe that we can set  $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4 = 0$ , provided we assume the simple relation

$$\Pi^*(K_{i,i+1}) = \tilde{P}_{i,i+1} = Q_{i,i+1}P_{i,i+1} , \quad (2.18)$$

and also use the two general conditions (1.6) satisfied by the  $Q$  matrix. Consequently, the projected monodromy matrix (2.14) would give us a novel realisation of extended  $Y(gl_M)$  Yangian algebra, if the spin dependent operator  $\tilde{P}_{i,i+1}$  occuring in the relation (2.9) is defined according to eqn.(2.18). It is worth noting that for the special case  $Q_{i,i+1} = \mathbb{1}$  one gets back  $\Pi^*(K_{i,i+1}) = P_{i,i+1}$ , which was used in ref.3 to find out a realisation of  $Y(gl_M)$  Yangian algebra through the conserved quantities of standard spin CS model (1.1). Now, by applying again the conditions (1.6) and well known properties of  $P_{i,i+1}$ , it is easy to verify that  $\tilde{P}_{i,i+1}$  operators defined by (2.18) indeed satisfy the permutation algebra (2.11). So, the form of a general  $\tilde{P}_{ij}$  (with  $j - i > 1$ ) can be obtained by using (2.12) and (2.18) as

$$\Pi^*(K_{ij}) = \tilde{P}_{ij} = (Q_{i,i+1}Q_{i,i+2} \cdots Q_{ij}) P_{ij} (Q_{j-1,i}Q_{j-2,i} \cdots Q_{i+1,i}) . \quad (2.19)$$

Thus our relations (2.18) and (2.19) give a general prescription of defining the projection operation  $\Pi^*$ , which can be used to construct the realisation (2.14) of extended  $Y(gl_M)$  algebra associated with any possible solution of YBE written in the form (1.5).

Next, we try to find out the spin CS Hamiltonian which would exhibit the extended  $Y(gl_M)$  Yangian symmetry and, therefore, commute with all elements of the  $T^0(u)$  matrix (2.14). For the special case  $Q_{0,0'} = \mathbb{1}$ , it is possible to derive such a spin CS Hamiltonian (1.1) from the quantum determinant of  $Y(gl_M)$  algebra [3]. However, it is difficult to obtain the quantum determinant of extended  $Y(gl_M)$  Yangian algebra associated with a rational solution (1.5), unless we take some specific form of the corresponding  $Q$  matrix.

So, we will describe here a rather adhoc procedure of constructing quantum determinant like objects, which would commute with all elements of  $T^0(u)$  (2.14) for any given choice of the related  $Q$  matrix. For this purpose we first define a set of operators  $I_n$  through the power series expansion:  $\prod_{i=1}^N (u - \hat{D}_i) = \sum_{n=0}^N I_n u^{N-n}$  and use eqn.(2.7) to find that

$$\left[ f(I_1, I_2, \dots, I_N), \hat{T}^0(u) \right] = 0, \quad \left[ f(I_1, I_2, \dots, I_N), \tilde{P}_{i,i+1} K_{i,i+1} \right] = 0, \quad (2.20)$$

where  $f(I_1, I_2, \dots, I_N)$  is an arbitrary polynomial function of  $I_n$ s and  $\hat{T}^0(u)$  is given by (2.8). Now, by applying (2.20) and (2.15), it is easy to see that  $\Pi^* [f(I_1, I_2, \dots, I_N)]$  will commute with all elements of the projected monodromy matrix (2.14). In particular, the choice  $\frac{2\pi^2}{L^2} \Pi^* (I_1^2 - I_2) = \frac{2\pi^2}{L^2} \Pi^* \left( \sum_{i=1}^N \hat{D}_i^2 \right)$  would give us such a Casimir operator, which can be written more explicitly by using eqn.(2.5) as

$$\tilde{H} = \frac{2\pi^2}{L^2} \Pi^* \left( \sum_{i=1}^N \hat{D}_i^2 \right) = -\frac{1}{2} \sum_{i=1}^N \left( \frac{\partial}{\partial x_i} \right)^2 + \frac{\pi^2}{L^2} \sum_{i < j} \frac{\beta(\beta + \tilde{P}_{ij})}{\sin^2 \frac{\pi}{L}(x_i - x_j)}. \quad (2.21)$$

Evidently the above expression can be interpreted as a spin CS Hamiltonian, where the operators  $\tilde{P}_{ij}$ , defined by eqns.(2.18) and (2.19), produce the spin dependent interactions. It is obvious that for the special case  $Q = \mathbb{1}$ , (2.21) reduces to the original spin CS Hamiltonian (1.1) containing only two-body spin dependent interactions. However, it is clear from eqn.(2.19) that,  $\tilde{P}_{ij}$  would generally depend on all spin variables associated with  $j - i + 1$  number of particles indexed by  $i, i + 1, \dots, j$ . So in contrast to the case of usual permutation operator  $P_{ij}$ , which acts nontrivially only on the spin spaces of  $i$ -th and  $j$ -th particles and represents a two-body interaction, the new operator  $\tilde{P}_{ij}$  would lead to a many-body spin dependent interaction in the Hamiltonian (2.21).

It should be mentioned that a spin CS Hamiltonian like (2.21) was studied earlier and solved exactly by applying a ‘generalised’ antisymmetric projection operator on the eigenfunctions of Dunkl operators [32]. However, in contrast to the present case, no method was prescribed in ref.32 about the way of constructing spin dependent operators  $\tilde{P}_{ij}$  from a given solution of YBE. So our analysis not only reveals a rich symmetry structure of the Hamiltonian (2.21), but also prescribes a very convenient method of constructing such Hamiltonian through the rational solution of YBE (1.5). In the following, we like to

briefly recall the procedure of solving the Hamiltonian (2.21) and show that the rational solution (1.5) also plays a crucial role in finding out the corresponding wave functions. So we make an ansatz for the wave function  $\tilde{\psi}$  of spin CS Hamiltonian (2.21) as [3,32]:  $\tilde{\psi}(x_1, \dots, x_N; \alpha_1, \dots, \alpha_N) = \left[ \prod_{i < j} \sin \frac{\pi}{L}(x_i - x_j) \right]^\beta \tilde{\phi}(x_1, \dots, x_N; \alpha_1, \dots, \alpha_N)$ , where it is assumed that  $\beta > 0$  to avoid singularity at the origin. Now, by applying the canonical commutation relations  $[\frac{\partial}{\partial x_j}, x_k] = \delta_{jk}$ , one may easily find that

$$\tilde{H}\tilde{\psi} = \frac{2\pi^2}{L^2} \left[ \prod_{i < j} \sin \frac{\pi}{L}(x_i - x_j) \right]^\beta \Pi_{(-)}^* (\mathcal{H}^*) \tilde{\phi}, \quad (2.22)$$

where  $\Pi_{(-)}^*$  is a projection operator defined by  $\Pi_{(-)}^* (K_{ij}) = -\tilde{P}_{ij}$ ,  $\mathcal{H}^* = \sum_{i=1}^N d_i^2$  and

$$d_i = z_i \frac{\partial}{\partial z_i} + \beta \left( i - \frac{N+1}{2} \right) - \beta \sum_{j>i} \theta_{ij} (K_{ij} - 1) + \beta \sum_{j<i} \theta_{ji} (K_{ij} - 1). \quad (2.23)$$

These  $d_i$ s may be considered as a ‘gauge transformed’ variant of the Dunkl operators  $\hat{D}_i$  (2.5) and they satisfy an algebra quite similar to (2.7). So one can construct the eigenvectors of  $\mathcal{H}^*$  by simultaneously diagonalising the mutually commuting set of operators  $d_i$ . To this end, however, it is helpful to make an ordering [3] of the corresponding basis elements characterised by the monomials like  $z_1^{\lambda_1} z_2^{\lambda_2} \dots z_N^{\lambda_N}$ , where  $\{\lambda_1, \dots, \lambda_N\} \equiv [\lambda]$  is a sequence of non-negative integers with homogeneity  $\lambda = \sum_{i=1}^N \lambda_i$ . Due to such ordering of monomials within a given homogeneity sector, it turns out that the operators  $d_i$  and  $\mathcal{H}^*$  can be represented through some simple block-triangular matrices. By taking advantage of this block-triangular property, it is not difficult to find that

$$\mathcal{H}^* \xi_{[\lambda]}(z_1, z_2, \dots, z_N) = \sum_{i=1}^N \left[ \lambda'_i - \beta \left( \frac{N+1}{2} - i \right) \right]^2 \xi_{[\lambda]}(z_1, z_2, \dots, z_N), \quad (2.24)$$

where  $[\lambda']$  is a permutation of sequence  $[\lambda]$  with the property  $\lambda'_1 \leq \lambda'_2 \leq \dots \leq \lambda'_N$ , and the asymmetric Jack polynomial  $\xi_{[\lambda]}(z_1, z_2, \dots, z_N)$  is a suitable linear combination of  $z_1^{\lambda_1} z_2^{\lambda_2} \dots z_N^{\lambda_N}$ , and other monomials of relatively lower orders. Though it is rather difficult to write down the general form of this  $\xi_{[\lambda]}$ , one can derive it easily for the case of low-lying excitations through diagonalisation of small block-triangular matrices. Furthermore, by using any given eigenfunction of  $\mathcal{H}^*$ , it is possible to construct a set of degenerate wave functions corresponding to the spin CS Hamiltonian (2.21) in the following way.

Let  $\rho(\alpha_1, \alpha_2, \dots, \alpha_N)$  be an arbitrary spin dependent function and  $\tilde{\Lambda}$  be a ‘generalised’ antisymmetric projection operator which satisfies the relation

$$\tilde{P}_{i,i+1} K_{i,i+1} \tilde{\Lambda} = - \tilde{\Lambda} , \quad (2.25)$$

for all  $i$ . With the help of eqns.(2.22), (2.24) and (2.25), one can prove that [32]

$$\tilde{\psi} = \left[ \prod_{i < j} \sin \frac{\pi}{L} (x_i - x_j) \right]^\beta \tilde{\Lambda} \left( \xi_{[\lambda]}(z_1, z_2, \dots, z_N) \rho(\alpha_1, \alpha_2, \dots, \alpha_N) \right) \quad (2.26)$$

would be an eigenfunction of the spin CS Hamiltonian (2.21) with eigenvalue

$$\epsilon_{[\lambda]} = \frac{2\pi^2}{L^2} \sum_{i=1}^N \left[ \lambda'_i - \beta \left( \frac{N+1}{2} - i \right) \right]^2 .$$

Since this eigenvalue does not depend on the choice of arbitrary function  $\rho(\alpha_1, \alpha_2, \dots, \alpha_N)$ , one usually gets a set of degenerate eigenfunctions through the relation (2.26).

Now, by combining (2.18) and (2.25), we see that the antisymmetric projector  $\tilde{\Lambda}$  satisfies the relation  $(Q_{i,i+1} P_{i,i+1} K_{i,i+1}) \tilde{\Lambda} = - \tilde{\Lambda}$ . So it is evident that the explicit form of  $\tilde{\Lambda}$  will depend on the choice of corresponding  $Q_{i,i+1}$  matrix. For example, in the simplest case of a spin CS model containing only two particles ( $N = 2$ ), we find that  $\tilde{\Lambda} = \mathbb{1} - K_{12} Q_{12} P_{12}$  satisfies the relation (2.25). By substituting this  $\tilde{\Lambda}$  to eqn.(2.26), we can write down the wave function for two particle spin CS Hamiltonian as

$$\tilde{\psi} = \left[ \sin \frac{\pi}{L} (x_1 - x_2) \right]^\beta (\mathbb{1} - K_{12} Q_{12} P_{12}) \left( \xi_{[\lambda]}(z_1, z_2) \rho(\alpha_1, \alpha_2) \right) . \quad (2.27)$$

It is curious to notice that the  $Q$  matrix, originally appeared in the solution (1.5) of YBE and the definition of extended  $Y(gl_M)$  algebra, also plays an important role in constructing the related wave function (2.27). So this  $Q$  matrix provides a direct link between the symmetry of spin CS models and their exact wave functions.

It is worth noting that, while deriving the results of this section, we have not used anywhere the power series expansion (1.7) which is valid only for the  $Q$  matrices associated with multi-parameter deformed  $Y(gl_M)$  Yangian algebra. Therefore, our results would be equally applicable for the case of multi-parameter deformed  $Y(gl_M)$  Yangian symmetries as well as their nonstandard variants. Thus, a spin CS Hamiltonian like (2.21) would

exhibit either of these two types of Yangian symmetries, depending on the specific choice of corresponding  $Q$  matrix.

### 3 Conserved quantities and Lax pairs of novel spin CS models

Though in the previous section we have seen that  $T^0(u)$  matrix (2.14) generates the conserved quantities of spin CS Hamiltonian (2.21), we have not yet derived those conserved quantities in an explicit way. Our aim here is to write down those conserved quantities in a compact form and find out their connection with the Lax pair of quantum integrable spin CS model (2.21). For this purpose, we like to recall first the procedure of constructing the conserved quantities of usual spin CS model (1.1) from the related Lax pair [8,3]. The Lax pair of CS model (1.1) consists of two  $N \times N$  dimensional matrices  $\mathcal{L}$  and  $\mathcal{M}$ , whose operator valued elements are given by

$$\mathcal{L}_{ij} = \delta_{ij} z_j \frac{\partial}{\partial z_j} + \beta (1 - \delta_{ij}) \theta_{ij} P_{ij}, \quad \mathcal{M}_{ij} = -2\beta' \delta_{ij} \sum_{k \neq i} h_{ik} P_{ik} + 2\beta' (1 - \delta_{ij}) h_{ij} P_{ij}, \quad (3.1a, b)$$

where  $\beta' = \frac{2\pi^2}{L^2} \beta$ ,  $\theta_{ij} = \frac{z_i}{z_i - z_j}$  and  $h_{ij} = \theta_{ij} \theta_{ji}$ . It should be observed that, unlike the case of previously discussed Lax operators (2.3) or (2.4), the above defined Lax pair does not depend on any auxiliary space and can not give a solution of QYBE in a straightforward fashion. However, through direct calculation it can be checked that, the Lax pair (3.1a,b) and the spin CS Hamiltonian (1.1) obey the relations

$$[H, \mathcal{L}_{ij}] = \sum_{k=1}^N (\mathcal{L}_{ik} \mathcal{M}_{kj} - \mathcal{M}_{ik} \mathcal{L}_{kj}), \quad [H, X_j^{\alpha\beta}] = \sum_{k=1}^N X_k^{\alpha\beta} \mathcal{M}_{kj}, \quad \sum_{k=1}^N \mathcal{M}_{jk} = 0, \quad (3.2a, b, c)$$

where  $X_i^{\alpha\beta}$  denotes a spin dependent operator which act as  $|\alpha\rangle\langle\beta|$  on the spins of  $i$ -th particle and leave all other particles untouched. By using eqn.(3.2a,b,c) one can interestingly

prove that the set of operators given by

$$T_n^{\alpha\beta} = \sum_{i,j=1}^N X_i^{\alpha\beta} (\mathcal{L}^n)_{ij} , \quad (3.3)$$

commute with the Hamiltonian (1.1).

The relation (3.3) gives an explicit expression for the conserved quantities of spin CS Hamiltonian (1.1) through the matrix elements of corresponding  $\mathcal{L}$  operator. However, from our discussions in the previous section, it is natural to expect that the  $Q = \mathbb{1}$  limit of  $T^0(u)$  matrix (2.14) should also produce these conserved quantities through some power series expansion. In fact it has been already shown that [3], the  $Q = \mathbb{1}$  limit of projected monodromy matrix (2.14) can be expanded as

$$T^0(u) = \mathbb{1} + \beta \sum_{n=0}^{\infty} \frac{1}{u^{n+1}} \sum_{\alpha,\beta=1}^M \left( X_0^{\alpha\beta} \otimes T_n^{\beta\alpha} \right) , \quad (3.4)$$

where  $T_n^{\alpha\beta}$ s are given by (3.3). At present we like to construct an analogue of eqn.(3.4) for a completely general  $Q$  matrix. Such a construction should yield the conserved quantities of spin CS Hamiltonian (2.21) and may also help to find out the related Lax pair. In this context it should be noted that, for proving the relation (3.4), a conjecture is made in ref.3 as

$$T^0(u) = \Pi \left( \mathbb{1} + \beta \sum_{i=1}^N \frac{P_{0i}}{u - D_i} \right) , \quad (3.5)$$

where  $\Pi(K_{ij}) = P_{ij}$ ,  $T^0(u)$  represents the  $Q = \mathbb{1}$  limit of our monodromy matrix (2.14) and  $D_i$ s are another type of Dunkl operators given by

$$D_i = z_i \frac{\partial}{\partial z_i} + \beta \sum_{j \neq i} \theta_{ij} K_{ij} . \quad (3.6)$$

It is easy to check that these  $D_i$ s satisfy the relations

$$K_{ij} D_i = D_j K_{ij} , \quad [K_{ij}, D_k] = 0 , \quad [D_i, D_j] = \beta (D_i - D_j) K_{ij} , \quad (3.7)$$

where  $k \neq i, j$ . We propose now a generalisation of the conjecture (3.5) as

$$T^0(u) = \Pi^* \left( \Omega + \beta \sum_{i=1}^N \frac{\mathbb{P}_{0i}}{u - D_i} \right) , \quad (3.8)$$

where  $T^0(u)$  is defined by (2.14), and

$$\Omega = Q_{01}Q_{02}\cdots Q_{0N}, \quad \mathbf{P}_{0i} = (Q_{01}Q_{02}\cdots Q_{0,i-1}) P_{0i} (Q_{0,i+1}Q_{0,i+2}\cdots Q_{0N}). \quad (3.9a, b)$$

By using the relations (1.6), (2.18) and (2.19), we have checked the validity of above conjecture for systems containing small number of particles. Moreover, it is evident that at the limit  $Q = \mathbb{1}$  (when one can put  $\Pi^* = \Pi$ ,  $\Omega = \mathbb{1}$ , and  $\mathbf{P}_{0i} = P_{0i}$ ), equation (3.8) reproduces the previous conjecture (3.5). So, in the following, we shall assume that (3.8) is a valid relation for any possible choice of corresponding  $Q$  matrix and all values of particle number  $N$ .

Next, we like to derive two relations which will be used shortly to express the conjecture (3.8) in a more convenient form. First of all, by applying eqns.(3.6), (3.7) and (2.9), one can show that

$$\Pi^*(D_i^n) = \sum_{j=1}^N (\tilde{\mathcal{L}}^n)_{ij}, \quad (3.10)$$

where  $\tilde{\mathcal{L}}$  is a  $N \times N$  matrix with elements given by

$$\tilde{\mathcal{L}}_{ij} = \delta_{ij} z_j \frac{\partial}{\partial z_j} + \beta (1 - \delta_{ij}) \theta_{ij} \tilde{P}_{ij}. \quad (3.11)$$

It may be noted that (3.10) is a straightforward generalisation of the known relation [3]:  $\Pi(D_i^n) = \sum_{j=1}^N (\mathcal{L}^n)_{ij}$ . Secondly, by using the standard relation:  $P_{0i} = \sum_{\alpha,\beta=1}^M X_0^{\alpha\beta} \otimes X_i^{\beta\alpha}$  and the conditions (1.6), we find that the operator  $\mathbf{P}_{0i}$  (3.9b) can be rewritten as

$$\mathbf{P}_{0i} = \sum_{\alpha,\beta=1}^M X_0^{\alpha\beta} \otimes \tilde{X}_i^{\beta\alpha}, \quad (3.12)$$

where

$$\tilde{X}_i^{\beta\alpha} = Q_{i,i+1}Q_{i,i+2}\cdots Q_{iN} X_i^{\beta\alpha} Q_{i1}Q_{i2}\cdots Q_{i,i-1}. \quad (3.13)$$

The expression (3.12) is more suitable for our purpose than (3.9b), since in (3.12) we have only one operator  $X_0^{\alpha\beta}$  which depends on the 0-th auxiliary space.

Now, with the help of eqns.(3.10) and (3.12), we find that the relation (3.8) can be expressed in a nice form

$$T^0(u) = \Omega + \beta \sum_{n=0}^{\infty} \frac{1}{u^{n+1}} \sum_{\alpha,\beta=1}^M (X_0^{\alpha\beta} \otimes \tilde{T}_n^{\beta\alpha}), \quad (3.14)$$

where  $\tilde{T}_n^{\alpha\beta}$ s are given by

$$\tilde{T}_n^{\alpha\beta} = \sum_{i,j=1}^N \tilde{X}_i^{\alpha\beta} (\tilde{\mathcal{L}}^n)_{ij} . \quad (3.15)$$

It is worth noting that eqns.(3.14) and (3.15) give us the desired generalisation of previous relations (3.4) and (3.3), for the case of an arbitrary  $Q$  matrix. So the operators  $\tilde{T}_n^{\alpha\beta}$  represent the conserved quantities of spin CS Hamiltonian (2.21), for any possible choice of corresponding  $Q$  matrix. It may also be observed that the operator  $\Omega$ , appearing in eqn.(3.14), would generate  $M^2$  number of additional conserved quantities (all of which become trivial at  $Q = \mathbb{1}$  limit). However, due to the fact that  $\Omega$  (3.9a) depends on the 0-th auxiliary space in a very complicated way, we find it difficult to write down the explicit form of these  $M^2$  number of additional conserved quantities.

It may be noticed that the above discussion, which yields the explicit form of conserved quantities (3.15), heavily depends on our conjecture (3.8). So, in the following, we like to show in an independent way that  $\tilde{T}_n^{\alpha\beta}$ s are indeed conserved quantities for the spin CS Hamiltonian (2.21). For this purpose we first compare the two expressions (3.3) and (3.15). Such comparison clearly indicates that the operator  $\tilde{\mathcal{L}}$  (3.11) may be treated as a generalisation of  $\mathcal{L}$  (3.1a), for the case of an arbitrary  $Q$  matrix. Moreover, it is evident that one can produce the matrix elements of  $\tilde{\mathcal{L}}$  from that of  $\mathcal{L}$  (3.1a), through the simple substitution:  $P_{ij} \rightarrow \tilde{P}_{ij}$ . So we make a similar substitution to eqn.(3.1b) and write down the matrix elements of corresponding  $\tilde{\mathcal{M}}$  as

$$\tilde{\mathcal{M}}_{ij} = -2\beta' \delta_{ij} \sum_{k \neq i} (h_{ik} \tilde{P}_{ik}) + 2\beta' (1 - \delta_{ij}) h_{ij} \tilde{P}_{ij} . \quad (3.16)$$

Now, we interestingly find that the four operators  $\tilde{H}$ ,  $\tilde{\mathcal{L}}$ ,  $\tilde{\mathcal{M}}$  and  $\tilde{X}_i^{\beta\alpha}$ , given by equations (2.21), (3.11), (3.16) and (3.13) respectively, satisfy the relations

$$[\tilde{H}, \tilde{\mathcal{L}}_{ij}] = \sum_{k=1}^N (\tilde{\mathcal{L}}_{ik} \tilde{\mathcal{M}}_{kj} - \tilde{\mathcal{M}}_{ik} \tilde{\mathcal{L}}_{kj}) , \quad (3.17a)$$

$$[\tilde{H}, \tilde{X}_j^{\alpha\beta}] = \sum_{k=1}^N \tilde{X}_k^{\alpha\beta} \tilde{\mathcal{M}}_{kj} , \quad \sum_{k=1}^N \tilde{\mathcal{M}}_{jk} = 0 . \quad (3.17b, c)$$

Again, these relations are a straightforward generalisation of the previous equation (3.2a,b,c). In fact the eqns.(3.2a,b,c) and (3.17a,b,c) are exactly same in form and related to each other through the substitutions  $\mathcal{L} \leftrightarrow \tilde{\mathcal{L}}$ ,  $\mathcal{M} \leftrightarrow \tilde{\mathcal{M}}$ ,  $X_k^{\alpha\beta} \leftrightarrow \tilde{X}_k^{\alpha\beta}$  and  $H \leftrightarrow \tilde{H}$ . So



it is clear that  $\tilde{\mathcal{L}}$  (3.11) and  $\tilde{\mathcal{M}}$  (3.16) represents the Lax pair associated with the spin CS Hamiltonian  $\tilde{H}$  (2.21). Furthermore, by using relations (3.17a,b,c), it is easy to directly check that the operators  $\tilde{T}_n^{\alpha\beta}$  (3.15) commute with  $\tilde{H}$ . Thus we are able to prove in an independent way that  $\tilde{T}_n^{\alpha\beta}$ s are the conserved quantities of spin CS Hamiltonian (2.21) and find out how these conserved quantities are related to the corresponding Lax pair.

## 4 Specific examples of spin CS models with extended $Y(gl_M)$ Yangian symmetry

In the previous sections we have developed a rather general framework for constructing the spin CS Hamiltonian which would exhibit an extended  $Y(gl_M)$  Yangian symmetry, and also found out the related Lax pair as well as conserved quantities. Let us derive now a few particular solutions of YBE which can be expressed in the form (1.5) and subsequently apply our general results to obtain the concrete form of corresponding spin CS models, Lax pairs etc.

*Case 1.*

To generate a rational  $R$  matrix of the form (1.5), one may use the well known spectral parameter independent solution of YBE (1.4) given by

$$R_{00'} = \sum_{\sigma=1}^M \epsilon_{\sigma}(q) e_{\sigma\sigma}^0 \otimes e_{\sigma\sigma}^{0'} + \sum_{\sigma \neq \gamma} e^{i\phi_{\gamma\sigma}} e_{\sigma\sigma}^0 \otimes e_{\gamma\gamma}^{0'} + (q - q^{-1}) \sum_{\sigma < \gamma} e_{\sigma\gamma}^0 \otimes e_{\gamma\sigma}^{0'}, \quad (4.1)$$

where  $e_{\sigma\gamma}^0$  is a basis operator on the 0-th auxiliary space with elements  $(e_{\sigma\gamma}^0)_{\tau\delta} = \delta_{\sigma\tau}\delta_{\gamma\delta}$ ,  $\phi_{\gamma\sigma}$ s are  $\frac{M(M-1)}{2}$  number of independent antisymmetric deformation parameters:  $\phi_{\gamma\sigma} = -\phi_{\sigma\gamma}$ , and each of the  $\epsilon_{\sigma}(q)$  can be freely taken as either  $q$  or  $-q^{-1}$  for any value of  $\sigma$ . In the special case when all  $\epsilon_{\sigma}(q)$ s take the same value (i.e., all of them are either  $q$  or  $-q^{-1}$ ), (4.1) can be obtained from the universal  $\mathcal{R}$ -matrix associated with  $U_q(sl(M))$  quantum group, for generic values of the parameter  $q$  [18-19]. On the other hand if  $\epsilon_{\sigma}(q)$ s do not take the same value for all  $\sigma$ , the corresponding ‘nonstandard’ solutions are found to be connected with the universal  $\mathcal{R}$  matrix of  $U_q(sl(M))$  quantum group, when  $q$  is

a root of unity [33-35]. It may also be noted that the parameters  $\phi_{\sigma\gamma}$  and  $\epsilon_\sigma(q)$  have appeared previously in the context of multi-parameter dependent quantisation of  $GL(M)$  group [36] and some asymmetric vertex models [37]. It is now easy to check directly that the  $R_{00'}$  matrix (4.1) satisfies the condition:  $R_{00'} - P_{00'}(R_{00'})^{-1}P_{00'} = (q - q^{-1})P_{00'}$ . So, by following the Yang-Baxterisation prescription [38] related to Hecke algebra, we can construct a spectral parameter dependent solution of YBE (1.4) as

$$R_{00'}(u) = q^{\frac{u}{\beta}} R_{00'} - q^{-\frac{u}{\beta}} P_{00'}(R_{00'})^{-1}P_{00'}. \quad (4.2)$$

Substituting the explicit form of  $R_{00'}$  (4.1) to the above expression, multiplying it by the constant  $\beta/(q - q^{-1})$  and subsequently taking the  $q \rightarrow 1$  limit, we get a rational solution of YBE in the form (1.5) where the corresponding  $Q$  matrix is given by

$$Q_{00'} = \sum_{\sigma=1}^M \epsilon_\sigma e_{\sigma\sigma}^0 \otimes e_{\sigma\sigma}^{0'} + \sum_{\sigma \neq \gamma} e^{i\phi_{\gamma\sigma}} e_{\sigma\sigma}^0 \otimes e_{\gamma\gamma}^{0'}, \quad (4.3)$$

$\epsilon_\sigma$ s being  $M$  number of discrete parameters, each of which can be freely chosen as 1 or  $-1$ . One may also verify directly that the above  $Q$  matrix satisfies the two required conditions (1.6). Consequently, the rational solution of YBE associated with this  $Q$  matrix can be used to define a class of extended  $Y(gl_M)$  Yangian algebra. Moreover, it is worth observing that only for the special choice  $\epsilon_\sigma = 1$  (or,  $\epsilon_\sigma = -1$ ) for all  $\sigma$ , the  $Q$  matrix (4.3) admits an expansion in the form (1.7). Therefore, only for these two choices of discrete parameters, the corresponding  $Q$  matrix generates a multi-parameter dependent deformation of  $Y(gl_M)$  Yangian algebra [21,22]. For all other choices of discrete parameters  $\epsilon_\sigma$ , we would get some nonstandard variants of  $Y(gl_M)$  Yangian algebra.

Next we substitute the specific form of  $Q$  matrix (4.3) to eqn.(2.18) and find that

$$\tilde{P}_{i,i+1} = \sum_{\sigma=1}^M \epsilon_\sigma e_{\sigma\sigma}^i \otimes e_{\sigma\sigma}^{i+1} + \sum_{\sigma \neq \gamma} e^{i\phi_{\gamma\sigma}} e_{\sigma\sigma}^i \otimes e_{\gamma\gamma}^{i+1}. \quad (4.4)$$

From our discussion in sec.2 it is evident that, any particular choice of  $\epsilon_\sigma$ s and  $\phi_{\gamma\sigma}$ s in the above expression of  $\tilde{P}_{i,i+1}$  would give us a representation of the permutation algebra (2.11). The action of  $\tilde{P}_{i,i+1}$  (4.4) on the space  $\mathcal{F}$  can easily be written as

$$\tilde{P}_{i,i+1} |\alpha_1 \alpha_2 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_N\rangle = \exp(i\phi_{\alpha_i \alpha_{i+1}}) |\alpha_1 \alpha_2 \cdots \alpha_{i+1} \alpha_i \cdots \alpha_N\rangle, \quad (4.5)$$

where we have used the notation  $e^{i\phi_{\sigma\sigma}} = \epsilon_{\sigma}$ . It is interesting to observe that, the above ‘anyon like’ representation of permutation algebra not only interchanges the spins of two particles but also picks up an appropriate phase factor. Moreover, by substituting (4.4) to (2.19) one can find out the operators  $\tilde{P}_{ij}$ , when  $j - i > 1$ . The action of such an operator on the space  $\mathcal{F}$  is given by

$$\begin{aligned} \tilde{P}_{ij} |\alpha_1 \alpha_2 \cdots \alpha_i \cdots \alpha_j \cdots \alpha_N\rangle = \\ \exp \left\{ i \phi_{\alpha_i \alpha_j} + i \sum_{\tau=1}^M n_{\tau} (\phi_{\tau \alpha_j} - \phi_{\tau \alpha_i}) \right\} |\alpha_1 \alpha_2 \cdots \alpha_j \cdots \alpha_i \cdots \alpha_N\rangle, \end{aligned} \quad (4.6)$$

where  $n_{\tau}$  denotes the number of times of occuring the particular spin orientation  $\tau$  in the configuration  $|\alpha_1 \cdots \alpha_i \cdots \alpha_p \cdots \alpha_j \cdots \alpha_N\rangle$ , when the index  $p$  in  $\alpha_p$  is varied from  $i + 1$  to  $j - 1$ . Thus, it turns out that the phase factor associated with the element  $\tilde{P}_{ij}$  actually depends on the spin configuration of  $(j - i + 1)$  number of particles. Consequently the operator  $\tilde{P}_{ij}$ , which acts nontrivially on the spin space of all these  $(j - i + 1)$  number of particles, would generate a highly nonlocal many-body spin dependent interaction in the CS Hamiltonian (2.21). Evidently at the special case  $\epsilon_{\sigma} = 1$  and  $\phi_{\gamma\sigma} = 0$  for all  $\sigma, \gamma$ , this  $\tilde{P}_{ij}$  reduces to two-body spin dependent interaction  $P_{ij}$  (2.13), which is used to define the usual spin CS model (1.1).

It may be noted that the ‘anyon like’ representations ((4.5),(4.6)) and the related spin CS Hamiltonians were considered earlier in ref.32. However, through our present analysis, we are able to construct these anyon like representations in a systematic way from the given solutions of YBE associated with  $Q$  matrix (4.3). Moreover we are able to show that the spin CS Hamiltonian (2.21), which contains these  $\tilde{P}_{ij}$  ((4.5),(4.6)) as spin dependent interaction, would exhibit the extended  $Y(gl_M)$  Yangian symmetry generated through  $Q$  matrix (4.3). Furthermore, by substituting (4.3), (4.5) and (4.6) to eqns.(3.11), (3.16) and (3.15), one can explicitly find out the corresponding Lax pair and conserved quantities.

It is important to note that, we can change the symmetry algebra of spin CS Hamiltonian (2.21) by tuning the discrete parameters  $\epsilon_{\sigma}$  and continuous parameters  $\phi_{\gamma\sigma}$  in the related  $Q$  matrix (4.3). Therefore, the study of corresponding degenerate wave functions should give us valuable information about the representation theory of a large class of ex-

tended Yangian algebras. In the following we like to construct the ground states of above considered spin CS models, for the simplest case when they contain only two spin- $\frac{1}{2}$  particles ( $N = M = 2$ ), and examine the dependence of these ground states on the related discrete as well as continuous deformation parameters. Indeed, by using eqn.(2.23), it is easy to see that the trivial monomial  $\xi(z_1, z_2) = 1$  would be an eigenvector of two Dunkl operators  $d_1, d_2$  and will also correspond to the lowest eigenvalue ( $\beta^2/2$ ) of operator  $\mathcal{H}^* = d_1^2 + d_2^2$ . Therefore, by substituting  $\xi(z_1, z_2) = 1$  to eqn.(2.27), the ground state associated with energy eigenvalue  $\frac{\pi^2\beta^2}{L^2}$  can be obtained as

$$\tilde{\psi} = \sin^\beta \left\{ \frac{\pi}{L}(x_1 - x_2) \right\} (1 - Q_{12}P_{12}) \rho(\alpha_1, \alpha_2). \quad (4.7)$$

Moreover, for spin- $\frac{1}{2}$  case, the arbitrary spin dependent function  $\rho(\alpha_1, \alpha_2)$  can be chosen in four different ways:  $|11\rangle, |12\rangle, |21\rangle$  and  $|22\rangle$ . Inserting these forms of  $\rho$  to eqn.(4.7) and also using eqn.(4.5) for  $N = M = 2$  case, we get three degenerate eigenfunctions like

$$\tilde{\psi}_1 = (1 - \epsilon_1) \Gamma^\beta |11\rangle, \quad \tilde{\psi}_2 = (1 - \epsilon_2) \Gamma^\beta |22\rangle, \quad \tilde{\psi}_3 = \Gamma^\beta (|12\rangle - e^{i\theta} |21\rangle), \quad (4.8a, b, c)$$

where  $\Gamma = \sin \left\{ \frac{\pi}{L}(x_1 - x_2) \right\}$  and  $\theta = \phi_{12}$ . Notice that the choice of  $\rho$  as  $|12\rangle$  or  $|21\rangle$  would lead to the same wave function  $\tilde{\psi}_3$  up to a multiplicative constant. Now it may be observed that, the substitution  $\epsilon_1 = \epsilon_2 = 1$  and  $\theta = 0$  to eqn.(4.8) would give us the ground state wave function of usual  $Y(gl_2)$  symmetric spin CS model (1.1) when  $N = M = 2$ . However, only the wave function  $\tilde{\psi}_3$  in eqn.(4.8) remains nontrivial after the above mentioned substitution, which in this case actually gives a nondegenerate ground state. In a similar way one finds a nondegenerate ground state even for the slightly different case:  $\epsilon_1 = \epsilon_2 = 1$  and  $\theta \neq 0$ . But, it should be noticed that the choice  $\epsilon_1 = \epsilon_2 = 1$  and  $\theta \neq 0$  leads to a spin CS Hamiltonian of type (2.21) whose symmetry algebra is given by a one parameter deformation of  $Y(gl_2)$  Yangian. Therefore, it is apparent that the change of usual  $Y(gl_2)$  Yangian symmetry of a spin CS model, through a continuous deformation parameter  $\theta$ , does not affect the degeneracy factor of related ground states. On the other hand if one substitutes  $\epsilon_1 = -\epsilon_2 = 1$  to eqn.(4.8), then both  $\tilde{\psi}_2$  and  $\tilde{\psi}_3$  would remain nontrivial and, as a result, we will get a doubly degenerate ground state. But, from our previous discussion it is known that the choice  $\epsilon_1 = -\epsilon_2 = 1$  in  $Q$  matrix

(4.3) leads to a nonstandard variant of  $Y(gl_2)$  Yangian. So we curiously find that, by switching over to a nonstandard variant of  $Y(gl_2)$  Yangian symmetry from its standard counterpart, one can change the degeneracy factor of the related ground states.

*Case 2.*

Recently, some new rational solutions of YBE is constructed from the universal  $\mathcal{R}$  matrices of deformed Yangian algebras [24]. In particular, an explicit solution is found in the form (1.5) where the corresponding  $Q$  matrix is given by

$$Q_{00'} = \mathbb{1} + 2\xi r_{00'} + 2\xi^2 (r_{00'})^2, \quad (4.9)$$

with

$$\begin{aligned} r_{00'} = & \frac{1}{2} \sum_{\sigma < M+1-\sigma} \left( h_{\sigma}^0 \otimes e_{\sigma, M+1-\sigma}^{0'} - e_{\sigma, M+1-\sigma}^0 \otimes h_{\sigma}^{0'} \right) \\ & + \sum_{\sigma < \gamma < M+1-\sigma} \left( e_{\sigma\gamma}^0 \otimes e_{\gamma, M+1-\sigma}^{0'} - e_{\gamma, M+1-\sigma}^0 \otimes e_{\sigma\gamma}^{0'} \right), \end{aligned}$$

and  $h_{\sigma}^0 = e_{\sigma\sigma}^0 - e_{M+1-\sigma, M+1-\sigma}^0$ . Since this  $Q$  matrix admits an expansion like (1.7), the corresponding rational solution of YBE would generate a single parameter dependent deformation of  $Y(gl_M)$  Yangian algebra. Again, by substituting the  $Q$  matrix (4.9) to eqns.(2.18) and (2.19), one can construct a new representation of permutation algebra (2.10). Evidently, this representation of permutation algebra can be used to find out a spin CS Hamiltonian like (2.21), which would exhibit the above mentioned single parameter deformed  $Y(gl_M)$  Yangian symmetry. However, it is rather difficult to explicitly write down such representation of permutation algebra due to its complicated nature, and we present here only the action of  $\tilde{P}_{12}$  for  $N = M = 2$  case:

$$\begin{aligned} \tilde{P}_{12} |11\rangle &= |11\rangle, \quad \tilde{P}_{12} |12\rangle = |21\rangle - \xi |11\rangle, \\ \tilde{P}_{12} |21\rangle &= |12\rangle + \xi |11\rangle, \quad \tilde{P}_{12} |22\rangle = |22\rangle + \xi |12\rangle - \xi |21\rangle + \xi^2 |11\rangle. \end{aligned} \quad (4.10)$$

Remarkably, this  $\tilde{P}_{12}$  can create new spin components which are not present in the original spin configuration. By substituting the  $Q$  matrix (4.9) and corresponding  $\tilde{P}_{ij}$  operators to eqns.(3.11), (3.16) and (3.15), in principle one can also find out the related Lax pair as well as conserved quantities. Moreover, by using eqns.(4.10) and (4.7), it is easy to

construct the ground state of associated spin CS model (2.21) for  $N = M = 2$  case as

$$\tilde{\psi} = \sin^\beta \left\{ \frac{\pi}{L} (x_1 - x_2) \right\} (|12\rangle - |21\rangle + 2\xi|11\rangle). \quad (4.11)$$

Thus we get here a nondegenerate ground state, which reduces to the ground state of usual  $Y(gl_2)$  symmetric spin CS model at  $\xi = 0$  limit. Thus we see again that, change of Yangian symmetry through a continuous deformation parameter does not affect the degeneracy of related ground state.

*Case 3.*

We propose another rational solution of YBE which can be expressed in the form (1.5), where the corresponding  $Q$  matrix is given by

$$Q_{00'} = \mathbb{1} + \xi \sum_{\sigma=2}^M (e_{\sigma\sigma}^0 \otimes e_{1\sigma}^{0'} - e_{1\sigma}^0 \otimes e_{\sigma\sigma}^{0'}). \quad (4.12)$$

It is clear that this  $Q$  matrix would generate a new type of single parameter deformed  $Y(gl_M)$  Yangian algebra. Moreover, by substituting this  $Q$  matrix to eqn.(2.18), it is possible to construct a representation of permutation algebra (2.11) as

$$\tilde{P}_{i,i+1} = P_{i,i+1} + \xi \sum_{\sigma=2}^M (e_{\sigma\sigma}^i \otimes e_{1\sigma}^{i+1} - e_{1\sigma}^i \otimes e_{\sigma\sigma}^{i+1}), \quad (4.13)$$

where  $e_{\sigma\gamma}^i \equiv X_i^{\sigma\gamma}$ . For the simplest  $N = M = 2$  case, the action of above permutation operator may be written as

$$\tilde{P}_{12}|11\rangle = |11\rangle, \tilde{P}_{12}|12\rangle = |21\rangle, \tilde{P}_{12}|21\rangle = |12\rangle, \tilde{P}_{12}|22\rangle = |22\rangle - \xi|12\rangle + \xi|21\rangle. \quad (4.14)$$

By substituting (4.13) to (2.19), one can also find out the operators  $\tilde{P}_{ij}$ , when  $j - i > 1$ . Evidently, these permutation operators will give us a spin CS Hamiltonian of the form (2.21), which would exhibit a deformed  $Y(gl_M)$  Yangian symmetry related to the  $Q$  matrix (4.12). Again, in principle, we can explicitly construct the Lax pair as well as conserved quantities for such spin CS Hamiltonian, by inserting the  $Q$  matrix (4.12) and associated  $\tilde{P}_{ij}$  operators to the general relations (3.11), (3.16) and (3.15). Moreover, with the help of eqns.(4.7) and (4.14), it is rather easy to see that the corresponding nondegenerate ground state (for  $N = M = 2$  case) would actually coincide with the ground state of usual  $Y(gl_2)$  symmetric spin CS model.

Case 4.

Finally we indicate about a particular class of possible  $Q$  matrix solutions, which using (1.5) yields rational solutions of YBE and through (2.18) constructs novel spin CS models (2.21). Such  $Q$  matrix solutions may be given as

$$Q_{i,i+1} = F_{i,i+1} F_{i+1,i}^{-1}, \quad (4.15)$$

where  $F_{i,i+1}$  are some representations of twisting operators defined in ref.39. Remarkably, the necessary conditions for the twisting operators can be shown also to be sufficient for the  $Q$ -matrix (4.15) as a solution of (1.6) [24]. Few concrete examples of such twisting matrices may be given as

$$\text{i) } F_{i,i+1} = \mathbb{1} + \xi \sigma_3^i \otimes \sigma_-^{i+1} \quad \text{and} \quad \text{ii) } F_{i,i+1} = \mathbb{1} + \xi \sum_{\alpha=2}^M e_{\alpha\alpha}^i \otimes e_{1\alpha}^{i+1}.$$

For some twisting operators the construction as

$$Q_{i,i+1} = F_{i,i+1} \Omega_{i,i+1} F_{i+1,i}^{-1}$$

with  $\Omega_{i,i+1} = \sum_{\sigma=1}^M \epsilon_{\sigma} e_{\sigma\sigma}^i \otimes e_{\sigma\sigma}^{i+1} + \sum_{\sigma \neq \gamma} e_{\sigma\sigma}^i \otimes e_{\gamma\gamma}^{i+1}$  may generate  $Q$ -matrices corresponding to the nonstandard Yangian algebras. Such an example of the twisting operator is

$$F_{ii+1} = \exp \left[ i \sum_{\sigma \neq \gamma} h_{\sigma}^i \otimes h_{\gamma}^{i+1} \phi_{\sigma\gamma} \right],$$

where  $h_{\sigma}^i = e_{\sigma\sigma}^i - e_{\sigma+1,\sigma+1}^i$  and  $\phi_{\sigma\gamma}$  are deforming parameters with  $\phi_{\gamma\sigma} = -\phi_{\sigma\gamma}$ .

## 5 Concluding Remarks

In this article we have constructed the general form of spin Calogero-Sutherland (CS) model which would satisfy the extended (i.e., multi-parameter dependent including non-standard variant of)  $Y(gl_M)$  Yangian symmetry. An important feature of such CS models is that they contain spin dependent many-body type interactions, which can be calculated directly from the associated rational solutions of Yang-Baxter equation. More

interestingly these spin dependent interactions can be expressed through some novel representations of permutation algebra on the combined internal space of all particles. We have also established the integrability by finding out the general forms of conserved quantities and Lax pairs for this class of spin CS models. As fruitful applications of the formalism we have constructed some concrete examples of spin CS models which exhibit the extended  $Y(gl_M)$  Yangian symmetry and discussed about the structure of the related ground state wave functions. Finally we have indicated the possible connections with twisting operators in some particular cases.

The existence of extended  $Y(gl_M)$  Yangian symmetry in the above mentioned class of spin CS models might lead to interesting applications in several directions. As it is well known, the degeneracy of wave functions for usual spin CS model can be explained quite nicely through the representations of  $Y(gl_M)$  algebra. So it is natural to expect that the representations of extended  $Y(gl_M)$  algebra would play a similar role in identifying the degenerate multiplets of corresponding spin CS models. Conversely, one may also be able to extract valuable information about the representations of extended  $Y(gl_M)$  Yangians, by studying the wave functions of associated spin CS models. In this article we have analysed the degeneracy of a few ground state wave functions, which indicated that the representations of nonstandard variants of  $Y(gl_M)$  Yangian algebra may differ considerably from their standard counterpart. In particular it has been found that the continuous deformation by multiparameters seems not to change the degeneracy pattern. However the nonstandard cases with discrete change of symmetries affect the degeneracy picture with a tendency of creating more degenerate states.

The representations of nonstandard variants of  $Y(gl_M)$  Yangian algebra might turn out to be a rather interesting subject for future investigation. Moreover, one may also try to use the extended  $Y(gl_M)$  symmetry in spin CS models for calculating their dynamical correlation functions and various thermodynamic properties. Finally, we hope that it would be possible to find out many other new type of quantum integrable systems with long range interactions, which would exhibit the extended  $Y(gl_M)$  Yangian symmetry.



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